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The Aversion Integral of Actuarial Risk Dynamics in Nigeria

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Abstract

The aim of this work is to find a new condition to compute aversion to risk integral which solves the associated second order differential equation with boundary conditions. The evaluation of an individual aversion to risk forms an integral part of expert's investment opinion. The evaluation technique of general insurance is theoretically deficient in formation and deepenedscientific methodologies. Investors' total wealth is usually categorizedinto assets which are assigned to short term project and free asset which assignment is subject to indefinite interval of time. The insufficient risk methodologies looks appropriate for the former category while the latter category is distributed in line with individual risk aversion intensity.

Keywords: Risk, Aversion, Differential Equation, Integral, Premium
JEL Codes: E10, E11

Introduction

This paper aims to establish a different condition to evaluate aversion to risk, aversion integral which the solves the associated second order differential equation with boundary conditions. Aversion is the mathematical construct of the feeling which guides an insurance agent taking a decision whose outcome is an uncertain event. Analytic and stochastic forms have been constructed through Taylor's expansion of utility about the initial wealth. In Thomas (2016), it was stated that the analytic technique of Taylor's expansion will only be valid when the share or interest in insurance business are conspicuously small and consequently the risk aversion will be close to zero for every one making its measurement difficult and subject to quantisation noise. Aversion to risk was originally introduced by in Pratt(1964) and Arrow(1963,1965,1970) where professional expertise was drawn to examine the association between behavior to risk and wealth. From Arrow (1970), it is apparent that the evaluation of this concept as wealth differs is strikingly observable to predict inference under risk uncertainties where it was suggested that aversion to risk is proportional to wealth. Empirical studies such as Holt and Laury (2002); Szpiro(1983); Eisenhauer and Halek (1999) all fall in line with the hypothesis of Arrow (1970) that aversion to risk increase with wealth. Bellante and Saba (1986) and levy (1994) however are among few other scholars who disagreed with Arrow (1970) based on empirical findings that aversion to risk increases with wealth where the authors discovered that aversion reduces as wealth declines. Aversion to risk may also be constant irrespective of the wealth level as seen in Szpiro (1986), Chiapori and Paiella (2011). Morin and Suarez (1983); Halek and Eisenhauer (2001) discovered that at low level of wealth, relative aversion to risk inreases with wealth but at higher levels of wealth, aversion decreases with wealth thus describing a novel non-linear association. The authors note that in case of the affluent individuals, the intensity of relative aversion to risk would be a weaker condition to the extent that the aversion co-efficient can be measured as constant. From the summary of discussions of the authors, one can easily infer that if ρ is the saturated point where u(w) is maximum so that $u'(\rho) = 0$ where the gradient function vanishes, then the risk aversion function $a(w) = -\frac{u''(w)}{u'(w)}$ proportionally grows with wealth and becomes asymptotically unbounded as $w \to \rho$. As the wealth approaches the saturation threshold ρ, then only a small fraction of utility is obtainable through a financial gain so that it will be financially unethical to assume further risk. Suppose function u(w) is concave, then the utility of the expected value of an uncertain amount of asset will be higher than the expected utility of the asset or wealth.

From the knowledge of numerical analysis u will be concave if when given any number x_0 in the interval (c, d) then there exists a constant parameter k depending on y_0 such
$u(y) - u(y_0) \leq k(y - y_0)$.
However, the utility function should meet the requirements of continuity in an interval (c, d) for it to be concave. If $u(y)$ is differentiable then $k = u'(y_0)$, but when $u(y)$ is not smooth, then there will be many k satisfying $u(y) - u(y_0) \leq k(y - y_0)$

Theorem
Assume the risk neutral is a linear combination of two different sub-risks $\bar{\Theta}_1$ and $\bar{\Theta}_2$, where $\bar{Y} = C_1\bar{\Theta}_1 + C_2\bar{\Theta}_2$, then $\frac{\partial^2\Pi(C_1, C_2)}{\partial C_2^2} = \left[\frac{\bar{\Theta}_2\sigma^2}{u'(w - \mu_Y)}\right]$ and

$$\frac{\partial^2\Pi(C_1, C_2)}{\partial C_1^2} = \left[\frac{\sigma^2_{\bar{\Theta}_1}}{u'(w - \mu_Y)}\right]$$

Proof

$$E(\bar{Y}) = E(C_1\bar{\Theta}_1 + C_2\bar{\Theta}_2) = C_1E(\bar{\Theta}_1) + C_2E(\bar{\Theta}_2)$$

$\sigma^2_{\bar{Y}} = C_1^2\sigma^2_{\bar{\Theta}_1} + C_2^2\sigma^2_{\bar{\Theta}_2} + 2C_1C_2COV(\bar{\Theta}_1, \bar{\Theta}_1)$, where $var(\bar{\Theta}_1) =$ σ2Θ1

$$(\mu_Y - \Sigma^+) = \left[\frac{\mu_2 u''(w - \mu_Y)}{2u'(w - \mu_Y)}\right] = \left[\frac{\left(C_1^2\sigma^2_{\bar{\Theta}_1} + C_2^2\sigma^2_{\bar{\Theta}_2} + 2C_1C_2COV(\bar{\Theta}_1, \bar{\Theta}_2)\right)u''(w - \mu_Y)}{2u'(w - \mu_Y)}\right]$$

Thus

$$\Pi(C_1, C_2) = \left[\frac{\left(C_1^2\sigma^2_{\bar{\Theta}_1} + C_2^2\sigma^2_{\bar{\Theta}_2} + 2C_1C_2COV(\bar{\Theta}_1, \bar{\Theta}_2)\right)u''(w - \mu_Y)}{2u'(w - \mu_Y)}\right]$$

$\frac{\partial\Pi(C_1, C_2)}{\partial C_1} = \left[\frac{(C_1\sigma^2_{\bar{\Theta}_1} + C_2COV(\bar{\Theta}_1, \bar{\Theta}_2))u''(w - \mu_Y)}{u'(w - \mu_Y)}\right]$. if we set $\frac{\partial\Pi(C_1, C_2)}{\partial C_1} = 0$,

$$\frac{\left(C_1\sigma^2_{\bar{\Theta}_1} + C_2COV(\bar{\Theta}_1, \bar{\Theta}_2)\right)u''(w - \mu_Y)}{u'(w - \mu_Y)} = 0$$

$$-\frac{u''(w - \mu_Y)}{u'(w - \mu_Y)} = \frac{C_1\sigma^2_{\bar{\Theta}_1}}{C_2COV(\bar{\Theta}_1, \bar{\Theta}_2)}$$

$\frac{2(\Sigma^+ - \mu_Y)}{\mu_2} = \left[-\frac{u''(w - \mu_Y)}{u'(w - \mu_Y)}\right]$. Thus the new aversion coefficient is defined as below

$$a(w, \mu_Y) = \frac{2(\Sigma^+ - \mu_Y)}{\mu_2} = \frac{C_1\sigma^2_{\bar{\Theta}_1}}{\left(C_2COV(\bar{\Theta}_1, \bar{\Theta}_2)\right)} \qquad Result 1$$

$$\Sigma^+ = \mu_Y + \frac{\mu_2 C_1\sigma^2_{\bar{\Theta}_1}}{2\left(C_2COV(\bar{\Theta}_1, \bar{\Theta}_2)\right)}$$

$$\lim_{C_1 \to 0}\frac{\partial\Pi(C_1, C_2)}{\partial C_1} =$$

$$\lim_{C_1\to 0}\left[\frac{\left(C_1\sigma^2_{\bar{\Theta}_1} + C_2 COV(\bar{\Theta}_1, \bar{\Theta}_2)\right)u''(w-\mu_Y)}{u'(w-\mu_Y)}\right]$$

$$\lim_{C_1\to 0}\frac{\partial\Pi(C_1, C_2)}{\partial C_1} = \frac{C_2 COV(\bar{\Theta}_1, \bar{\Theta}_2)u''(w-\mu_Y)}{u'(w-\mu_Y)} = -\frac{C_2 COV(\bar{\Theta}_1, \bar{\Theta}_2)C_1\sigma^2_{\bar{\Theta}_1}}{C_2 COV(\bar{\Theta}_1, \bar{\Theta}_2)}$$

$$\lim_{C_1\to 0}\frac{\partial\Pi(C_1, C_2)}{\partial C_1} = -\sigma^2 C_1\bar{\Theta}_1 \text{ or}$$

$$\lim_{C_1\to 0}\frac{\partial\Pi(C_1, C_2)}{\partial C_1} = \frac{-2(\Sigma^+ - \mu_Y)C_2 COV(\bar{\Theta}_1, \bar{\Theta}_2)}{\mu_2}$$

Taking the second derivative, we have

$$\frac{\partial^2\Pi(C_1, C_2)}{\partial {C_1}^2} = \left[\frac{\sigma^2_{\bar{\Theta}_1}}{u'(w-\mu_Y)}\right]$$

$$\frac{\partial\Pi(C_1, C_2)}{\partial C_2} = \left[\frac{\left(C_2\sigma^2_{\bar{\Theta}_2} + C_1 COV(\bar{\Theta}_1, \bar{\Theta}_2)\right)u''(w-\mu_Y)}{u'(w-\mu_Y)}\right]$$

$$\lim_{C_1\to 0}\frac{\partial\Pi(C_1, C_2)}{\partial C_2} = \lim_{C_1\to 0}\left[\frac{\left(C_2\sigma^2_{\bar{\Theta}_2} + C_1 COV(\bar{\Theta}_1\bar{\Theta}_2)\right)u''(w-\mu_Y)}{u'(w-\mu_Y)}\right]$$

$$\lim_{C_1\to 0}\frac{\partial\Pi(C_1, C_2)}{\partial C_2} = \frac{C_2\sigma^2_{\bar{\Theta}_2}}{u'(w-\mu_Y)}$$

$$\frac{\partial^2\Pi(C_1, C_2)}{\partial {C_2}^2} = \left[\frac{\bar{\Theta}_2\sigma^2}{u'(w-\mu_Y)}\right]$$

Theorem

If $\bar{Y} = C_1\bar{\Theta}_1 + C_2\bar{\Theta}_2$ with $E(\bar{\Theta}_1) = E(\bar{\Theta}_2) = 0$**,** then

$$\frac{\partial\Pi(C_1, C_2)}{\partial C_1} = 0 \text{ and } \frac{\partial\Pi(C_1, C_2)}{\partial C_2} = 0$$

Proof

$$\bar{Y} = C_1\bar{\Theta}_1 + C_2\bar{\Theta}_2 \text{ with } E(\bar{\Theta}_1) = E(\bar{\Theta}_2) = 0$$

Recall that the risk premiumΠ must satisfy the condition that

$Eu(w + \bar{Y}) = u(w - \Pi)$ and hence

$$Eu(w + C_1\bar{\Theta}_1 + C_2\bar{\Theta}_2) = u(w - \Pi(C_1, C_2))$$

Differentiating both sides we have

$$\frac{\partial Eu(w + C_1\bar{\Theta}_1 + C_2\bar{\Theta}_2)}{\partial C_1} = \frac{\partial u(w - \Pi(C_1, C_2))}{\partial C_1}$$

$$\frac{\partial Eu(w + C_1\bar{\Theta}_1 + C_2\bar{\Theta}_2)}{\partial C_2} = \frac{\partial u(w - \Pi(C_1, C_2))}{\partial C_2}$$

Since expectation and differentiation operators can be swapped then

$$\frac{E\,\partial u(w + C_1\bar{\Theta}_1 + C_2\bar{\Theta}_2)}{\partial C_1} = \frac{\partial u(w - \Pi(C_1, C_2))}{\partial C_1} \text{ and}$$

$$\frac{E\,\partial u(w + C_1\bar{\Theta}_1 + C_2\bar{\Theta}_2)}{\partial C_2} = \frac{\partial u(w - \Pi(C_1, C_2))}{\partial C_2}$$

$$\frac{E(\bar{\Theta}_1)\,\partial u(w + C_1\bar{\Theta}_1 + C_2\bar{\Theta}_2)}{\partial C_1} = -\frac{\partial u(w - \Pi(C_1, C_2))}{\partial C_1}\frac{\partial\Pi(C_1, C_2)}{\partial C_1} \text{ and}$$

$$\frac{E(\bar{\Theta}_2)\,\partial u(w + C_1\bar{\Theta}_1 + C_2\bar{\Theta}_2)}{\partial C_2} = -\frac{\partial u(w - \Pi(C_1, C_2))}{\partial C_2}\frac{\partial\Pi(C_1, C_2)}{\partial C_2}$$

$$\frac{E(\bar{\Theta}_1)\,\partial u(w + C_1\bar{\Theta}_1 + C_2\bar{\Theta}_2)}{\partial C_1} = -\frac{\partial u(w - \Pi(C_1, C_2))\left[\frac{\left(C_1\sigma^2_{\bar{\Theta}_1} + C_2 COV(\bar{\Theta}_1, \bar{\Theta}_2)\right)u''(w-\mu_Y)}{u'(w-\mu_Y)}\right]}{\partial C_1}$$

and

$$\frac{E(\bar{\Theta}_2)\,\partial u(w + C_1\bar{\Theta}_1 + C_2\bar{\Theta}_2)}{\partial C_2} = -\frac{\partial u(w - \Pi(C_1, C_2))}{\partial C_2}\left[\frac{\left(C_2\sigma^2_{\bar{\Theta}_2} + C_1 COV(\bar{\Theta}_1, \bar{\Theta}_2)\right)u''(w-\mu_Y)}{u'(w-\mu_Y)}\right]$$

By the conditions$E(\bar{\Theta}_1) = 0$ and $E(\bar{\Theta}_2) = 0$

$$\left[\frac{\left(C_1\sigma^2_{\bar{\Theta}_1} + C_2 COV(\bar{\Theta}_1, \bar{\Theta}_2)\right)u''(w-\mu_Y)}{u'(w-\mu_Y)}\right] = 0$$

$$\left[\frac{\left(C_2\sigma^2_{\bar{\Theta}_2} + C_1 COV(\bar{\Theta}_1, \bar{\Theta}_2)\right)u''(w-\mu_Y)}{u'(w-\mu_Y)}\right] = 0,$$

$$\frac{\partial\Pi(C_1, C_2)}{\partial C_1} = 0 \text{ and } \frac{\partial\Pi(C_1, C_2)}{\partial C_2} = 0$$

Suppose the limiting process above fallswithin the neighbourhoodof small risks, it may be inferredthat insurance companyhaving high absolute risk aversion $a(w)$will not be willing to cover small risks, because the minimum expected payoff $E(w - Y)$which will make the risk acceptable will be larger. It is clear in Clark, Frijters and Shields (2008);Deprez and Gerber (1985) that $a(w)$is a measure of the degree of risk aversion of the insurance agent and consequently from actuarial point of view $a(w)$ is a measure of the degreeof concavity of the utility function where the instantaneous speed at which marginal utilityis decreasing that is$u'' < 0$ is evaluated.When utility preferences elicitsprudence, income uncertainties reduces current consumption and saving increases. It is prudence that informs individuals to treat future uncertain income with utmost care not expend as much currently as they would if future income is certain. The saving which results from the knowledge that the future is randomly uncertain isthe precautionary saving measure. The income received after retirement is a direct application of prudence.In Leland (1968), precautionary saving requires convex marginal utility in addition to aversion to risk. Kimball (1990);Kimball and Weil (2009) proposed forces of intensity about precautionary saving motive, a measure of absolute prudence and relative prudence

$$a'(t) = \frac{u'(t)u'''(t) - u''(t)u''(t)}{[u'(w)]^2}$$

$$a'(t) = \frac{u'(t)u'''(t) - u''(t)u''(t)}{[u'(t)]^2}$$

$$= \frac{u'(t)u'''(t) - [u''(t)]^2}{[u'(t)]^2}$$

At maximum, $a'(t) = 0$ and $u'(t)u'''(t) - [u''(t)]^2 = 0$

$$u'(t)u'''(t) = u''(t)u''(t)$$

$$\frac{u'''(t)}{u''(t)} = \frac{u''(t)}{u'(t)}$$

$p(t) = -\frac{u'''(t)}{u''(t)}$is called absolute prudence and $A(t) = \frac{p(t)}{t}$ is the relative prudence substituting $u''(t) = -u'(t)a(t)$ in $a'(t)$, we have

$$a'(t) = \frac{u'(t)u'''(t) - [u''(t)]^2}{[u'(t)]^2}$$

$$= \frac{u'(t)u'''(t) - [a(t)]^2[u'(t)]^2}{[u'(t)]^2}$$

$$= \frac{u'''(t) - u'(t)[a(t)]^2}{u'(t)}$$

Again, $a'(t) = 0$, then $u'''(t) - u'(t)[a(t)]^2 = 0$

$$u'''(t) = u'(t)[a(t)]^2$$

$$a(t) = \pm\sqrt{\frac{u'''(t)}{u'(t)}} = -\sqrt{\frac{u'''(t)}{u'(t)}}$$

$=$ provided $u'''(t) > 0$

If $a'(t) < 0$, *then* $u'(t)u'''(t) - [u''(t)]^2 < 0 \Rightarrow u'(t)u'''(t) < [u''(t)]^2$

It is worthy of note that the policy holder's utility is a decreasing function if for every risk, the amount of premium with which he would exchange the risk for insurance coverage is relatively higher than his assets. The condition just described involves the third derivative of utility function especially when $a'(t)$ is less than zero.

Aversion to Risk among Insurance Agents with Same Assets

Suppose $u_n$is a concave transformation of function u_i , i = 1,2,…, (n−1): ∃an increasing and concave function $f(u)$with $\frac{df}{du} > 0$ and $\frac{d^2f}{du^2} \leq 0$such that for all w

$$u_j(w) = f(u_1, u_2, u_3, u_4, \dots, u_{n-2}, u_{n-1}), a_{u_j}(w) \geq a_{u_i}(w)$$

We assume that there are n insurance agents,$u_1, u_2, u_3, u_4, \dots, u_n$who have the same arbitrary asset w. An insurance agent

u_1 will be more risk-averse than any another agent u_2, insurance agent u_2 will be more risk-averse than any another agent u_3, Insurance agent u_i will be more risk-averse than any another agent u_j for $j = i + 1$ and $i \neq j, i = 1, 2, 3, 4, \ldots, (n-1)$ with the same initial asset if given any risk which is undesirable for the insurance agent u_i is also undesirable for insurance agent u_j, so that the risk premium of any risk Y is bigger for any other insurance agent u_j than for insurance agent u_i. The condition holds independently of the common initial asset level w of the insurance agents. If the condition above is limited to small risks then it is required that

$$\frac{-u_n''(w)}{u_n'(w)} \geq -\frac{u_{n-1}''(w)}{u_{n-1}'(w)} \geq -\frac{u_{n-2}''(w)}{u_{n-2}'(w)} \geq \ldots \geq -\frac{u_2''(w)}{u_2'(w)} \geq \frac{-u_1''(w)}{u_1'(w)}$$

$$a_{u_n}(w) \geq a_{u_{n-1}}(w) \geq a_{u_{n-2}}(w) \geq \ldots \geq a_{u_2}(w) \geq a_{u_1}(w)$$

for all w. Suppose that the insurance agents are restricted to small risks, then u_j is more risk-averse than u_i if function $a_{u_j}(w)$ is uniformly bigger than $a_{u_i}(w)$ so that u_j will be more concave than u_i by reason of aversion to risk. This condition means that $u_j(w)$ is a concave transformation of $u_i(w)$ and consequently ∃an increasing and concave function f such that

$$u_n(w) = f(u_1, u_2, u_3, u_4, \ldots, u_{n-2}, u_{n-1})$$

$$\frac{\partial u_n(w)}{\partial w} = \frac{\partial f}{\partial u_1}\frac{\partial u_1}{\partial w} + \frac{\partial f}{\partial u_2}\frac{\partial u_2}{\partial w} + \frac{\partial f}{\partial u_3}\frac{\partial u_3}{\partial w} + \frac{\partial f}{\partial u_{n-2}}\frac{\partial u_{n-2}}{\partial w} + \ldots + \frac{\partial f}{\partial u_{n-1}}\frac{\partial u_{n-1}}{\partial w} \quad (1)$$

$$\frac{\partial^2 u_n(w)}{\partial w^2} = \left[\frac{\partial u_1}{\partial w}\right]^2 \frac{\partial^2 f}{\partial u_1^2} + \frac{\partial f}{\partial u_1}\frac{\partial^2 u_1(w)}{\partial w^2} + \left[\frac{\partial u_2}{\partial w}\right]^2 \frac{\partial^2 f}{\partial u_2^2} + \frac{\partial f}{\partial u_2}\frac{\partial^2 u_2(w)}{\partial w^2} + \left[\frac{\partial u_3}{\partial w}\right]^2 \frac{\partial^2 f}{\partial u_3^2} + \frac{\partial f}{\partial u_3}\frac{\partial^2 u_3(w)}{\partial w^2} + \left[\frac{\partial u_4}{\partial w}\right]^2 \frac{\partial^2 f}{\partial u_4^2} + \frac{\partial f}{\partial u_4}\frac{\partial^2 u_4(w)}{\partial w^2} + \left[\frac{\partial u_{n-2}}{\partial w}\right]^2 \frac{\partial^2 f}{\partial u_{n-2}^2} + \frac{\partial f}{\partial u_{n-2}}\frac{\partial^2 u_{n-2}(w)}{\partial w^2} + \cdots + \left[\frac{\partial u_{n-1}}{\partial w}\right]^2 \frac{\partial^2 f}{\partial u_{n-1}^2} + \frac{\partial f}{\partial u_{n-1}}\frac{\partial^2 u_{n-1}(w)}{\partial w^2} \quad (2)$$

Dividing (2) by(1) when taking $n = 2$ and by mutual comparison we have

$$\frac{d \log_e \left[\frac{\partial u_2(w)}{\partial w}\right]}{dx} = \left[\frac{\partial u_1}{\partial w}\right] \frac{d \log_e \left[\frac{\partial f}{\partial u_1}\right]}{du_1} + \frac{d \log_e \left[\frac{\partial u_1}{\partial w}\right]}{dw}$$

$$a_{u_2}(w) = a_{u_1}(w) + \left[\frac{\partial u_1}{\partial w}\right] \frac{d \log_e \left[\frac{\partial f}{\partial u_1}\right]}{du_1},$$

$$a_{u_n}(w) \geq a_{u_{n-1}}(w)$$

$$a_{u_{n-1}}(w) \geq a_{u_{n-2}}(w)$$

.
.
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$$a_{u_2}(w) \geq a_{u_1}(w)$$

The Aversion Integral

Now $a(w) = -\frac{\frac{du'(w)}{dw}}{\frac{du(w)}{dw}}$ implies

$-\int_0^w \frac{d\log_e u'(z)}{dz} = \int_0^w a(z)\,dz$. This is called the aversion integral hence

$$\log_e u'(w) - \log_e u'(0) = -\int_0^w a(t)\,dt$$

$$\int_0^w a(t)\,dt = \lim \sum a(t)\,\Delta t_k = \log_e u'(0) - \log_e u'(w) > 0$$

this is the risk aversion integral which converges and represents the difference of logarithm of utility function computed at two different points in the interval [0 w].

Now, $\int_0^w a(t)\,dt$ represents the total area under the curve of intensity $a(t)$ for $0 \le t \le w$ and can be numerically evaluated simply by invoking Newton-cotes formula with $2n$ number of strips and $(2n+1)$ number of $a(t)$ values. The risk aversion integral is the length of width $(W-0)$, multiplying the mean value of the aversion co-efficient function within the interval $(0,w)$, $\int_0^w a(t)\,dt = wf(\alpha), 0 < \alpha < w$. The average value of the aversion coefficient $a(t)$ is the sum of the weighted functional values divided by the sum of the weights where the weights are

$$W = W_0 + W_1 + W_2 + \ldots + W_n \text{ and } w = 2nh, \quad W_0 = 1, W_1 = 4, W_2 = 2, \ldots$$

$$W = 1 + 4 + 2 + 4 + 2 + 4 + 2 + 4 + \cdots + 1, \text{up to } (2n+1) \text{ times}$$

$$2 + 4 + 2 + 4 + 2 + 4 + \cdots \quad (2n)\text{times} = 2 + 2 + 2 + 2 + \ldots(\text{n times}) + 4 + 4 + 4 + 4 + \cdots..(\text{n times}) = \left\{\sum_1^n 2\right\} + \left\{\sum_1^n 4\right\} = 2n + 4n = 6n$$

Thus dividing the interval $(0,w)$ by $2n$ strips, we have

$$\int_0^w a(t)\,dt = \frac{h}{3}\{a_1 + 4a_2 + 2a_3 + 4a_4 + 2a_5 + \cdots + 4a_{n-2} + 2a_{n-1} + 4a_n + a_{n+1}\} - \frac{w(h^4)f^{(4)}(t)}{180}, 0 \le t \le w,$$

where a_i are the functional values of aversion co-efficient at the i-th point $w_n = nh$

$$\frac{w}{6n} = \frac{h}{3}$$

$\log_e u'(0) - \log_e u'(w) = \frac{h}{3}\{a_1 +$ 4a2+ 2a3+ 4a4+ 2a5+…+4an−2+ 2an−1+4an+ an+1 − w(h4)f4(t)180implies that

$$\{a_1 + 4a_2 + 2a_3 + 4a_4 + 2a_5 + \cdots + 4a_{n-2} + 2a_{n-1} + 4a_n + a_{n+1}\} - \frac{w(h^4)f^{(4)}(t)}{180}. = \log_e\left\{\frac{(u'(0))}{(u'(w)}\right\}^{\frac{3}{h}}$$

$$2\sum a_i + 4\sum a_i + a_1 - \frac{w(h^4)f^{(4)}(t)}{180} = \log_e\left\{\frac{(u'(0))}{(u'(w)}\right\}^{\frac{3}{h}}$$

From above, we have $\int_0^w a(t)\,dt = \log_e\left\{\frac{(u'(0))}{(u'(w)}\right\} \Rightarrow \left\{\frac{(u'(0)}{(u'(w)}\right\} = e^{\int_0^w a(t)dt}$ given that $u'(.) \neq o(1)$ that is, does not vanish.

In the trivial case where $u'(0) = 1$, (14) now becomes,

$u'(w) = e^{-\int_0^w a(t)dw}$. This is the gradient function for the utility functional u(.), again from (13),

$$u'(w) = e^{-\int_0^w a(t)dw} = 1 - \int_0^w a(t)\,dw$$

$$\int_0^w a(t)\,dt = -\int_0^w \frac{(u''(t)}{(u'(t)}\,dt$$

$$-\int_0^w a(t)(u'(t)\,dt = \int_0^w (u''(t)\,dt = u'(w) - u'(0)$$

$$u'(w) = u'(0) - \int_0^w a(t)(u'(t)\,dt \quad ,$$

integration by part yields,

$$u'(w) = u'(0) - \left\{a(w)u(w) - a(0)u(0) - \int_0^w u(t)a'(t)\,dt\right\}$$

$u'(w) = u'(0) + a(0)u(0) + \int_0^w a'(t)u'(t)\,dt - a(w)u(w).$

$a(w) = \frac{u'(0) + a(0)u(0) + \int_0^w a'(t)u'(t)dt - u'(w)}{u(w)},$

$u(w) > 0$

This establishes another form of risk aversion function. Therefore

$$u'(0) + a(0)u(0) + \int_0^w a'(t)u'(t)\,dt - a(w)u(w) = 1 - \int_0^w a(t)\,dw$$

Solutions to the Second Order Linear Differential Equations

Whenever the risk aversion function $a(w)$is given and it is required to determine the corresponding utility function, then we find a function $u(w)$which will satisfy the equation $G(w,u,u',u'') = 0$that is $u'' = g(w,u,u')$and such that $u(\pi) = 0$, $u'(\pi) = 1$ are the boundary conditions

$$a(w) = -\frac{u''(w)}{u'(w)} = -d\log_e u'(w)$$

$$a(w) = -d\log_e u'(w)$$

$$\int_\pi^t a(w)dw = -\int_\pi^t d\log_e u'(w)\ dw = -[\log_e u'(t) - \log_e u'(\pi)] = -[\log_e u'(t)] = \text{colog}_e u'(t)$$

$$u'(t) = e^{-\int_\pi^t a(w)dw}$$

$u(t) = \int_\pi^w e^{-\int_\pi^t a(w)dw}dt$so that

$u(w) = \int_\pi^w \left(e^{-\int_\pi^t a(w)dw}\right)dt$ for $(w - \pi > 0,$

$$u(w) = \int_\pi^w \left(e^{-\int_\pi^t k dw}\right)dt = \int_\pi^w e^{k\pi - kt}\ dt = \frac{e^{k\pi - kt}}{-k}\Big|_{t=\pi}^{t=w}$$

$$u(w) = \frac{e^{k\pi - k\pi}}{k} - \frac{e^{k\pi - kw}}{k} = \frac{1}{k} - \frac{e^{k\pi - kw}}{k} = \frac{1}{k} - \frac{1}{k}\left(e^{k\pi - kw}\right)$$

$$u(w) = \frac{1}{k} - \frac{1}{k}(1 + k\pi - kw) = \frac{1}{k} - \frac{1}{k} - \pi + w = w - \pi$$

$u(w + \pi) = w$, so $u'(w) = u'(w + \pi)$ implies $(w + \pi)$is equivalent to w

So fork ≥ 0, $u(w) = w - \pi$

If aversion $a(w) = k$ where k is a constant, then a change in assets may not necessarily result in a corresponding change in preference among risk. Again from the earlier discussions above

Theorem: The differential equation is solvable $u''(w) + a(w)u'(w) = 0$, under the boundary condition $u(\pi) = 0$ and $u'(\pi) = 1$, where $\pi \in [0, w]$

Recall that maximum premium $\Sigma^+ = \mu_Y + \frac{\mu_2}{2} a(w)$, assuming real constants μ_Y, μ_2 and Σ^+ . This second order differential equation has a two-parameter family of solutions. To obtain a unique solution, we invoke the standard boundary conditions.

$$2u'(w)\mu_Y - \mu_2 u''(w) = 2u'(w)\Sigma^+$$

$\sigma^2{}_Y u''(w) + 2[\Sigma^+ - \mu_Y]u'(w) = 0$subject to the following standard boundary conditions $u(\pi) = 0$ and $u'(\pi) = 1$, where $\pi \in [0, w]$

$$u''(w) + \frac{2[\Sigma^+ - \mu_Y]}{\mu_2} u'(w) = 0$$

The auxiliary equation is $s^2 + \frac{2[\Sigma^+ - \mu_Y]}{\mu_2} s = 0$

$$s\left[s + \frac{2[\Sigma^+ - \mu_Y]}{\sigma^2{}_Y}\right] = 0$$

$$s = 0 \text{ or } s = \frac{2[\mu_Y - \Sigma^+]}{\mu_2}$$

$$u(w) = k\,e^{\frac{2[\mu_Y - \Sigma^+]w}{\mu_2}}$$

$$u(\pi) = 0$$

$$k\,e^{\frac{2[\mu_Y - \Sigma^+]\pi}{\mu_2}} = 0 \text{and} k = 0$$

$$u'(w) = k\frac{2[\mu_Y - \Sigma^+]}{\mu_2} e^{\frac{2[\mu_Y - \Sigma^+]w}{\mu_2}}$$

$$u'(\pi) = k\frac{2[\mu_Y - \Sigma^+]}{\mu_2} e^{\frac{2[\mu_Y - \Sigma^+]\pi}{\mu_2}} = 1$$

$$K = \frac{\sigma^2{}_Y}{2[\mu_Y - \Sigma^+]} e^{\frac{-2[\mu_Y - \Sigma^+]\pi}{\mu_2}}$$

$$u(w) = \frac{\sigma^2{}_Y}{2[\mu_Y - \Sigma^+]} e^{\frac{2[\mu_Y - \Sigma^+]w - 2[\mu_Y - \Sigma^+]\pi}{\mu_2}}, \mu_Y > \Sigma^+ \quad \text{result2},$$

Reduction of Order

In actuarial literature, we are permitted to reduce the order of the aversion so as to apply a simple numerical procedure using the transformation

$$u'(w) = f(w)$$

$$u''(w) = f'(w)$$

$$a(w) = -\frac{f'(w)}{f(w)}$$

$$f(t+y) = f(y) + t\frac{df(t+y)}{dy} + \frac{t^2}{2!}\frac{d^2f(t+y)}{dy^2} + \frac{t^3}{3!}\frac{d^3f(t+y)}{dy^3} + \frac{t^4}{4!}\frac{d^4f(t+y)}{dy^4}$$

$$f(y-2) = f(y) - 2\frac{df(y)}{dy} + 2\frac{d^2f(y)}{dy^2} + \frac{4}{3}\frac{d^3f(y)}{dy^3} + \frac{2}{3}\frac{d^4f(y)}{dy^4}$$

$$f(y+2) = f(y) + 2\frac{df(y)}{dy} + 2\frac{d^2f(y)}{dy^2} + \frac{4}{3}\frac{d^3f(y)}{dy^3} + \frac{2}{3}\frac{d^4f(y)}{dy^4}$$

$$f(y-1) = f(y) - \frac{df(y)}{dy} + \frac{1}{2}\frac{d^2f(y)}{dy^2} - \frac{1}{6}\frac{d^3f(y)}{dy^3} + \frac{1}{24}\frac{d^4f(y)}{dy^4}$$

$$f(y+1) = f(y) + \frac{df(y)}{dy} + \frac{1}{2}\frac{d^2f(y)}{dy^2} + \frac{1}{6}\frac{d^3f(y)}{dy^3} + \frac{1}{24}\frac{d^4f(y)}{dy^4}$$

$$f(y-2)-f(y+2) = -\left[4\frac{df(y)}{dy} + \frac{8}{3}\frac{d^3f(y)}{dy^3}\right]$$

$$f(y-1)-f(y+1) = -\left[2\frac{df(y)}{dy} + \frac{1}{3}\frac{d^3f(y)}{dy^3}\right]$$

$$8f(y-1)-8f(y+1) = -\left[16\frac{df(y)}{dy} + \frac{8}{3}\frac{d^3f(y)}{dy^3}\right]$$

$$f(y-2)-f(y+2) - [8f(y-1) - 8f(y+1)] = -\left[4\frac{df(y)}{dy} + \frac{8}{3}\frac{d^3f(y)}{dy^3}\right] - \left[-16\frac{df(y)}{dy} - \frac{8}{3}\frac{d^3f(y)}{dy^3}\right]$$

$$f(y-2)-(y+2) - 8f(y-1) + 8f(y+1) = -4\frac{df(y)}{dy} - \frac{8}{3}\frac{d^3f(y)}{dy^3} + 16\frac{df(y)}{dy} + \frac{8}{3}\frac{d^3f(y)}{dy^3}$$

$$f(y-2)-f(y+2) - 8f(y-1) + 8f(y+1) = 12\frac{df(y)}{dy}$$

$$f(y-2)-f(y+2) - 8f(y-1) + 8f(y+1) = 12\frac{df(y)}{dy}$$

$$\frac{df(y)}{dy} = \frac{f(y+2) - f(y+2) - 8f(y-1) + 8f(y+1)}{12}$$ but

by the reduction of order

$$a(y) = -\frac{f'(y)}{f(y)}$$

$$= \frac{u'(y+2) - u'(y-2) + 8u'(y-1) - 8u'(y+1)}{12u'(y)}$$

Effect of Non Actuarially Neutral Risk $\bar{\mathbf{Y}}$

The condition arises when $E(Y) \neq \mu_Y$, that is when the insurance premium is not equal to the expected claim.

$$E[u(w-Y)] = E[u(w-\mu_Y)] + u'(w-\mu_Y)E(Y-\mu_Y) + \frac{u''(w-\mu_Y)E(Y-\mu_Y)^2}{2}$$

$$u(W-\Sigma^+) = u(w-\mu_Y) + u'(w-\mu_Y)(\mu_Y - \Sigma^+)$$

$$E[u(w-\mu_Y)] + au'(w-\mu_Y)E[(Y-\mu_Y)] + \frac{au''(w-\mu_Y)E(Y-\mu_Y)^2}{2} = u(w-\mu_Y) + u'(w-\mu_Y)(\mu_Y - \Sigma^+)$$

$$u(w-\mu_Y) + u'(w-\mu_Y)E[(Y-\mu_Y)] + \frac{u''(w-\mu_Y)\sigma^2{}_Y}{2} = u(w-\mu_Y) + u'(w-\mu_Y)(\mu_Y - \Sigma^+)$$

$$u'(w-\mu_Y)E[(Y-\mu_Y)] + \frac{u''(w-\mu_Y)\sigma^2{}_Y}{2} = u'(w-\mu_Y)(\mu_Y - \Sigma^+)$$

$$E[(Y-\mu_Y)] + \frac{u''(w-\mu_Y)\sigma^2{}_Y}{2u'(w-\mu_Y)} = (\mu_Y - \Sigma^+)$$

$$\Sigma^+ = \mu_Y - E[(Y-\mu_Y)] - \frac{u''(w-\mu_Y)\sigma^2{}_Y}{2u'(w-\mu_Y)} \qquad \text{result3}$$

.

Thus the risk premium is just enough to offset the actuarial value $E[(Y-\mu_Y)]$ of a small risk premium $\bar{Y}$ where the policy holder will be indifferent between having $\bar{Y}$ and not having it when the actuarial value is proportionately $a(w)$ times half the variance of $\bar{Y}$

Conclusions

In this paper, structural model have been developed which is built upon the existing work by incorporating assumptions such as that the aversion co-efficient has a convergent integral, that is $\int a(y)dy < \infty$ and that u(.) possess derivatives of all orders.Meaningful mathematical risk construct describing the likelihood for evidence based utility calculations can be arrived having the potential for new actuarial insight as a consequence. Risk aversion is the parameter describing the amount of satisfaction or utility preference derivable from money or goods. The conventional way of estimating aversion co-efficient when applied to purchase insurance employs Taylor's series expansion about the utility of the initial wealth. The implication here is that there should be infinitesimally small deviation from the wealth so that it will not be necessary to evaluate the structural form of the utility and as a result, there is no need to cross-check whether the aversion to risk has been manipulated when utility of two outcomes are contrasted. The Risk aversion derived here through analytic method is a function of insurance maximum premium, individual wealth and probability of loss.Actuaries suggest that the degree of decision maker response as a function of aversion intensity to a positive change in relative wealth provided that the absolute wealth remains constant depends on the accompanying change in the weightings assigned to the relative wealth and consequently a constant weight may cause aversion to risk intensity to increase. However an increase in the weightings assigned to relative wealth will reduce the aversion to risk. This paper finds applications in insurance of self protection for increased risk aversion, evaluating utility preferences of inequality aversion and aversion preferences in equilibrium asset pricing model.

Competing Interests

All authors have declared that there is no competing interest

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